SCHOOL OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

BACHELOR OF SCIENCE HONOURS DEGREE IN MATHEMATICS

LEVEL 4 SEMESTER 2

BACHELOR OF SCIENCE HONOURS DEGREE IN STATISTICS AND OPERATIONS RESEARCH

LEVEL 4 SEMESTER 1

MAIN EXAMINATION QUESTION PAPER

MODULE CODE	**HMAT/HSOR418**
MODULE NARRATION	**STOCHASTIC DIFFERENTIAL EQUATIONS**
DATE	**NOVEMBER 2024**
DURATION	**3 HOURS**

INSTRUCTIONS TO CANDIDATES:

1. Candidates may attempt **ANY FOUR** questions. Each question carries 25 Marks.

A1. (a) Define filtration. [2]

(b) Let $\{\mathcal{F}_i\}_{i \in I}$ be a family of $\sigma-$algebras on Ω. Prove that

$$\mathcal{F} = \bigcap \{\mathcal{F}_i\}_{i \in I}$$

is again a $\sigma-$algebra. [6]

(c) Solve the following stochastic differential equation, where X_t is the mean-reverting Ornstein-Uhlenbeck process,

$$dX_t = (m - X_t)dt + \sigma dB_t,$$

where m, σ are real constants and $B_t \in \mathbf{R}$. [5]

(d) Let μ be a measure defined on a $\sigma-$algebra $\mathcal{F}$. Show that if $A, B \in \mathcal{F}$ and $A \subseteq B$ with $\mu(B) < +\infty$, then $\mu(A) \leq \mu(B)$. [4]

(e) By using Ito's formula, show that

$$Z_t = \exp\left[\int_0^t \theta(s,\omega)dB_s - \frac{1}{2}\int_0^t \theta^2(s,\omega)ds\right]$$

is a martingale. [8]

A2. (a) State the properties of the Ito integral. [4]

(b) Assuming $B_0 = 0$, prove that

$$\int_0^t B_s dB_s = \frac{1}{2}{B_t}^2 - \frac{1}{2}t.$$

[6]

(c) State and prove the Martingale Representation Theorem. [15]

A3. (a) Check whether the process

$$X_t = B_t + 4t$$

is a martingale with respect to $\{\mathcal{F}_t\}$. [4]

(b) Solve

(i) $dX_t = rX_t dt + X_t\left(\sum_{k=1}^{n} \alpha_k dB_k(t)\right); X_0 > 0,$ [5]

(ii) $dX_t = \frac{1}{X_t}dt + \alpha X_t dB_t,$ [8]

(iii) $dX_t = {X_t}^{\gamma}dt + \alpha X_t dB_t.$ [8]

A4. Suppose the biomass of a forest at time t is given by

$$dX(t) = \mu dt + \sigma dB(t) + \theta \int_{\mathbf{R}} z\tilde{N}(dt, dz), \quad X(0) > 0,$$

where $\mu > 0, \delta > 0, \theta > 0$ are constants and we assume that $z \leq 0$ a.s.ν. At times $0 \leq \tau_1 < \tau_2 < \tau_3 < \ldots$ we decide to cut down the forest and replant it, with the cost $c + \tilde{X}(\tau_k{}^-)$ with $\tilde{X}(\tau_k{}^-) = X(\tau_k{}^-)\Delta_N X(\tau_k)$ where $c > 0, \lambda \in [0, 1]$ are constants and $\Delta_N X(\tau_k)$ is the jump in X at τ_k coming from the jump in $N(\tau_k, .)$ only, and not from the intervention. Find the sequence of stopping times $v = (\tau_1, \tau_2, \tau_3, \ldots)$ which maximizes the expected total dicounted net profit $J^v(s, x)$ given by

$$J^v(s,x) = E^x \left[\sum_{k=1}^{\infty} e^{-\rho(s+\tau_k)} (\tilde{X}(\tau_k{}^-) - C - \lambda(\tilde{X}(\tau_k{}^-)) \right]$$

where $\rho > 0$ is a given discounting exponent. [25]

A5. Find a stopping time τ that maximizes

$$E^{(s,x)}[e^{-\rho t}(X_t - a)] = E^{(s,x)}[g(\tau, X_\tau)],$$

where $g(t, \xi) = e^{-\rho t}(\xi - a)$. [25]

END OF QUESTION PAPER