# SCHOOL OF NATURAL SCIENCES

## DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

## BACHELOR OF SCIENCE HONOURS DEGREE IN STATISTICS AND OPERATIONS RESEARCH

## LEVEL 4 SEMESTER 2

## EXAMINATION QUESTION PAPER

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| MODULE CODE | HSOR 424 / HSOR401 / HMAT401 |
| MODULE NARRATION | ADVANCED PROBABILITY THEORY |
| DATE | |
| DURATION | 3 HOURS |

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**INSTRUCTIONS TO CANDIDATES:**

Candidates may attempt **ANY FOUR** questions.

**A1.** (a) For any $\epsilon > 0$, give an example of an irreducible Markov chain on a countably infinite state space, such that $|p_{ij} - p_{ik}| \leq \epsilon$ for all states $i; j$ and $k$. [10]

(b) Given Markov chain transition probabilities $\{P_{ij}\}_{i,j \in S}$ on a state space $S$, call a subset $C \subseteq S$ closed if $\sum_{j \in C} p_{ij} = 1$ for each $i \in C$. Prove that a Markov chain is irreducible if and only if it has no closed subsets (aside from the empty set and $S$ itself). [15]

**A2.** (a) Suppose $\Omega = \{1, 2, 3\}$ and $\mathcal{F}$ is a collection of all subsets of $\Omega$. Find (with proof), the necessary and sufficient conditions on the real numbers $x, y$ and $z$ such that there exists a countably additive probability measure $P$ on $\mathcal{F}$, with $x = P\{1, 2\}$, $y = P\{2, 3\}$ and $z = P\{1, 3\}$. [5]

(b) Suppose that $\Omega = \aleph$, and $P$ is defined for all $A \subseteq \Omega$ by $P(A) = |A|$ if $A$ is finite (where $|A|$ is the number of elements in the subset $A$) and $P(A) = \infty$ if $A$ is infinite. This $P$ is of course not a probability measure (in fact it is a counting measure), however we can still ask the following.

(i) Is $P$ infinitely additive? [5]

(ii) Is $P$ countably additive? [5]

(c) Let $\mathcal{F}_1; \mathcal{F}_2; \ldots$ be a sequence of collections of subsets of $\Omega$ such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for each $n$.

(i) Suppose that each $\mathcal{F}_i$ is an algebra. Prove that $U_{i=1}^{\infty} \mathcal{F}_i$ is also an algebra. [5]

(ii) Suppose that each $\mathcal{F}_i$ is a $\sigma$-algebra. Show (by counter example) that $U_{i=1}^{\infty} \mathcal{F}_i$ might not be a $\sigma$-algebra. [5]

**A3.** (a) Consider infinite, independent, fair coin tossing, and let $H_n$ be the event that the $n^{th}$ coin is head. Determine the following probabilities.

(i) $P(\bigcap_{i=1}^{9} H_{n+i} \ i.o)$, [3]

(ii) $P(\bigcap_{i=1}^{n} H_{n+i} \ i.o)$, [4]

(iii) $P(\bigcap_{i=1}^{[2 \log_2 n]} H_{n+i} \ i.o)$, [4]

(iv) $P(\bigcap_{i=1}^{[\log_2 n]} H_{n+i} \ i.o)$. [4]

(b) Suppose $\{A_n\} \nearrow A$. Let $f : \Omega \to \Re$ be any function. Prove that

$$\lim_{n \to \infty} \inf_{w \in A_n} f(w) = \inf_{w \in A} f(w).$$

[10]

**A4.** Consider the Markov chain with state space $S = \{1, 2, 3\}$ and transition probabilities $p_{12} = p_{23} = p_{31} = 1$. Let $\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$.

(a) Determine whether or not the chain is irreducible. [5]

(b) Determine whether or not the chain is aperiodic. [5]

(c) Determine whether or not the chain is reversible with respect to $\{\pi_i\}$. [5]

(d) Determine whether or not $\{\pi_i\}$ is a stationary distribution. [5]

(e) Determine whether or not

$$\lim_{n \to \infty} p_{11}^{(n)} = \pi_1.$$

[5]

**A5.** (a) Give (with proof) an example of two discrete random variables having the same mean and the same variance, but which are not identically distributed. [9]

(b) Prove that if $\{X_n\}$ converges to $X$ almost surely, then for each $\epsilon > 0$ we have $P(|X_n - X| \geq \epsilon i : o :) = 0)$. [6]

(c) Compute $E(X); E(X^2)$ and $V\ ar(X)$, where the law of $X$ is given by

(i) $\mathcal{L}(X) = \frac{1}{2}\delta_1 + \frac{1}{2}\lambda$, where $\lambda$ is Lebesgue measure on $[0, 1]$. [5]

(ii) $\mathcal{L}(X) = \frac{1}{3}\delta_2 + \frac{2}{3}\mu_N$, where $\mu_N$ is the standard normal distribution $N(0; 1)$. [5]

**END OF QUESTION PAPER**