GREAT ZIMBABWE UNIVERSITY

HSOR 421

FACULTY OF AGRICULTURE AND NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

BSc Honours in Statistics and Operations Research: Part 4 Semester 1

EXAMINATION: MAIN PAPER

HSOR 421: OPERATIONS RESEARCH TECHNIQUES

DATE: 2023

Time: 3 hours

Candidates may attempt **ANY FOUR** questions.

SECTION A

A1.

(a) Let

$$f(x_1; x_2) = 2x_1 - x_2 - x_1^2 + x_1x_2 - x_2^2.$$

Show that the function $f(x_1;x_2)$ has global maximum. [5]

(b) Investigate the stationary points of

$$f(x,y) = x^3 + y^3 - 3x - 12y + 20.$$

[5]

(c) Use the Lagrange method to optimise

$$f(\mathrm{X}) = x_2^2 + x_1^2 - 2x_1$$

subject to

$$x_1^2 + x_2^2 = 1.$$

(5)

(d) The Cobb-Douglas Production function for a production manufacturer is givenby $f(x_1, x_2) = 100x_1^{\frac{3}{4}}x_2^{\frac{1}{4}}$, where x_1 represents the unit of labour at $150 pe unit and x_2 represents the units of capital at $250 per unit. The total cost of labour and capital is limited to $50000. Find the maximum production level foı this manufacturer.

10]

A2. (a) What are the necessary conditions for optimality in constrained optimization? [4]

(b) Consider the problem,

Maximize

$$z = f(x) = 2x_1 + 4x_2 + x_1^2 + 3x_1x_2 + 4$$

Subject to:

$$-x_1 + 4x_2 \leq 4$$

$$6x_1 + 2x_2 \leq 1$$

$$x_1 - x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

(i) Write down the Kuhn-Tucker conditions for the problem. [6]

(ii) Verify that the optimal solution is $x_1 = x_2 = \lambda_1 = \lambda_2 = \lambda_3 = 0, \lambda_4 = 2, \lambda_5 = 4$. [10]

(iii) Show that this is the only solution to the problem. [5]

A3. (a) (i)Determine the definiteness of the symmetric 4 × 4 matrix.

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$$

(6)

(ii) Write down the Bellman equation for the value function of MDP (2)

(b) Minimize $f(x_1,x_1) = (x_1 - 3)^2 + 9(x_2 - 5)^2$, using the method of Conjugate gradients. Use the initial point $X_0 = (1,1)$. [10]

(c) Is $f(x_1; x_2; x_3) = x_1 + x_2^2 + x_3^3 - x_1x_2 - 3x_3$ convex or concave? [7]

A4.

(a) Perform four iterations using the steepest descent method to minimize

$$f(\mathbf{X}) = x_1^2 + 2x_1x_2 + 3x_2^2,$$

starting with $\mathbf{X}^{(0)} = (2,1)^T, \Delta y = 0.10.$ [10]

(b) Use the Lagrange method to optimise

$$f(x,y,z) = x^2 + y^2 + z^2$$

subject to $x + 2y + 3z = 14$.

[5]

(c) Maximize $y = x^2 \sin x$, $0 \leq x \leq \pi$ using a 14-point dichotomous search ($K = 7, \epsilon = 0.0001$ radians).

[10]

A5. (i) What is the optimal policy for an MDP? (2)

(ii) How can linear programming be used to find the optimal policy for an MDP? (3)

(a) The profit obtained by producing X_1 units of product A and X_2 units of B is approximated by the model

$$f(x_1, x_2) = 10x_1 + 8x_2 - (0.001)(x_1^2 + x_1x_2 + x_2^2) - 10000$$

Find the production level that produces the maximum profit. [10]

(b) Given linearly constrained convex programming problem:

Maximize

$$f(x_1, x_2) = 3x_1 + 4x_2 - x_1^3 - 2x_2^2,$$

subject to

$$x_1 + x_2 \leq 1,$$

$$x_1, x_2 \geq 0.$$

Starting from the initial trial solution $(x_1, x_2) = (\frac{1}{4}, \frac{1}{4})$, apply three iterations of the Frank-Wolfe algorithm. [10]

A6 (a) How can the Kuhn-Tucker conditions be used to solve a constrained optimization problem? (5)

(b) Write down the Kuhn-Tucker conditions for the following problem:

Maximize $f(x, y) = x^2 + y^2$

Subject to $x + y \leq 1$ (8)

(c) Define the following terms:

(i) state, action,

(ii)transition function,

(iii) reward function, and

(iv)discount factor. (8)

(d) Write down the mathematical model for an MDP (4)