SCHOOL OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

BACHELOR OF SCIENCE HONOURS DEGREE IN STATISTICS AND OPERATIONS RESEARCH

LEVEL 2 SEMESTER 1

EXAMINATION QUESTION PAPER

MODULE CODE **HSOR 2114**

MODULE NARRATION **PROBABILITY THEORY II**

DATE

DURATION **3 HOURS**

INSTRUCTIONS TO CANDIDATES:

1. Candidates may attempt **ALL** questions in Section A and ANY TWO questions from Section B

SECTION A

A1. Consider the gambling policies model, with $p = \frac{1}{3}; a = 6$ and $c = 8$.

(a) Compute the probability $s_{c;p}(a)$ that the player will win (i.e. hit c before hitting 0) if they bet 1 Dollar each time(that is, if $W_n \equiv 1$). [5]

(b) Compute the probability that the player will win if they bet 2 Dollars each time (i.e. if $W_n \equiv 2$). [5]

(c) Compute the probability that the player will win if they employ the strategy of Bold play(i.e., if $W_n = \min(X_{n-1}; c - X_{n-1})$). [10]

A2. (a) For any $\epsilon > 0$, give an example of an irreducible Markov chain on a countably infinite state space, such that $|p_{ij} - p_{ik}| \leq \epsilon$ for all states $i; j$ and k. [9]

(b) Given Markov chain transition probabilities $\{P_{ij}\}_{i,j \in S}$ on a state space S, call a subset $C \subseteq S$ closed if $\sum_{j \in C} p_{ij} = 1$ for each $i \in C$. Prove that a Markov chain is irreducible if and only if it has no closed subsets (aside from the empty set and S itself). [11]

SECTION B

B3. (a) Consider infinite, independent, fair coin tossing, and let H_n be the event that the n^{th} coin is head. Determine the following probabilities.

(i) $P(\bigcap_{i=1}^{9} H_{n+i} \ i.o)$, [4]

(ii) $P(\bigcap_{i=1}^{n} H_{n+i} \ i.o)$, [5]

(iii) $P(\bigcap_{i=1}^{[2\log_2 n]} H_{n+i} \ i.o)$, [5]

(iv) $P(\bigcap_{i=1}^{[\log_2 n]} H_{n+i} \ i.o)$. [5]

(b) Suppose $\{A_n\} \nearrow A$. Let $f : \Omega \to \Re$ be any function. Prove that

$$\lim_{n\to\infty} \inf_{w\in A_n} f(w) = \inf_{w \in A} f(w).$$

[11]

B4. Let $f(x) = ax^2 + bx + c$ be a second degree polynomial function (where $a, b, c \in \Re$ are constants).

(a) Find necessary and sufficient conditions on a, b and c such that the equation $E(f(\alpha X)) = \alpha^2 E(f(X))$ holds for all $\alpha \in \Re$ and all random variables X. [9]

(b) Find necessary and sufficient conditions on a, b and c such that the equation $E(f(X)) = E(f(X - \beta))$ holds for all $\beta \in \Re$ and all random variables X. [9]

(c) Do parts (*a*) and (*b*) account for the properties of the variance function? Explain. [12]

B5. Consider the Markov chain with state space $S = \{1, 2, 3\}$ and transition probabilities $p_{12} = p_{23} = p_{31} = 1$. Let $\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$.

(a) Determine whether or not the chain is irreducible. [6]

(b) Determine whether or not the chain is aperiodic. [6]

(c) Determine whether or not the chain is reversible with respect to $\{\pi_i\}$. [6]

(d) Determine whether or not $\{\pi_i\}$ is a stationary distribution. [6]

(e) Determine whether or not

$$\lim_{n\to\infty} p_{11}^{(n)} = \pi_1.$$

[6]

END OF QUESTION PAPER