GARY MAGADZIRE SCHOOL OF AGRICULTURE AND NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

BACHELOR OF SCIENCE HONOURS DEGREE IN STATISTICS AND OPERATIONS RESEARCH

LEVEL 4 SEMESTER 1/ 4 SEMESTER 2

EXAMINATION QUESTION PAPER

MODULE CODE	**HSOR 423**
MODULE NARRATION	**STOCHASTIC PROCESS 2**
DATE	
DURATION	**3 HOURS**

INSTRUCTIONS TO CANDIDATES:

1. Candidates to select 4 questions out of 5 given

Great Zimbabwe University
BSc Statistics and Operations Research
Examination: Advanced Stochastic Processes
Time: 3 Hours
Total Marks: 125

Answer any 4 questions out of the 5 provided. Each question carries 25 marks.

Question 1

Brownian motion (or Wiener process) is one of the most important stochastic processes in finance and physics.

(a) Define a standard Brownian motion $W(t)$. State its properties, including mean, variance, and the independent increments property. (8 marks)

(b) Show that for any $t > 0$, the random variable $W(t)$ is normally distributed with mean 0 and variance t. (6 marks)

(c) A particle undergoes Brownian motion with $W(t)$ representing its position at time t. If the particle starts at the origin, what is the probability that it will be at position 2 at time $t = 4$? (6 marks)

(d) Discuss the use of Brownian motion in modeling stock prices and financial markets. (5 marks)

Question 2

The Black-Scholes model is widely used to price options in financial markets. The stock price $S(t)$ is modeled as a stochastic process given by the stochastic differential equation (SDE):

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dW(t),$$

where μ is the drift, σ is the volatility, and $W(t)$ is standard Brownian motion.

(a) Explain the meaning of each term in the Black-Scholes SDE and provide a brief derivation of the Black-Scholes formula for the price of a European call option. (10 marks)

(b) Solve the SDE for $S(t)$ by using Itô's Lemma and show that $S(t)$ has a lognormal distribution. (8 marks)

(c) Consider an option with strike price \$50. The current stock price is \$55, the volatility is 20%, and the risk-free rate is 5%. Compute the price of a European call option with 6 months to maturity using the Black-Scholes formula. (7 marks)

Question 3

Stochastic Differential Equations (SDEs) are used to model systems affected by random noise.

(a) Define a Stochastic Differential Equation (SDE) and explain the difference between ordinary differential equations (ODEs) and SDEs. (6 marks)

(b) Consider the SDE:

$$dX(t) = \theta(\mu - X(t))\,dt + \sigma\,dW(t),$$

which is the Ornstein-Uhlenbeck process. Solve this SDE by using the method of integrating factors. (10 marks)

(c) Discuss how the Ornstein-Uhlenbeck process is used to model interest rates and mean-reverting processes in finance. (4 marks)

(d) Suppose the interest rate is modeled by the above SDE with parameters $\theta = 0.5$, $\mu = 0.03$, and $\sigma = 0.01$. Calculate the expected value and variance of $X(t)$ at time $t = 1$. (5 marks)

Question 4

Consider a production line where the number of defects in a day follows a Poisson process with a rate $\lambda = 4$ defects per day.

(a) Define a Poisson process and derive the formula for the probability of observing exactly k events (defects) in time t. (8 marks)

(b) Compute the probability that exactly 6 defects are found in a single day. (5 marks)

(c) Calculate the expected number of defects over a 5-day period. (5 marks)

(d) Explain how the Poisson process can be used to model customer arrivals at a service center and suggest ways to use this information to reduce waiting times. (7 marks)

Question 5

A factory has a machine repair station where machines arrive according to a Poisson process with an arrival rate of 2 machines per hour, and the service time follows an exponential distribution with a mean service rate of 3 machines per hour. This forms an $M/M/1$ queue.

(a) Derive the expression for the average number of machines in the system (both in the queue and being serviced). (8 marks)

(b) Compute the average waiting time for a machine before it is serviced. (6 marks)

(c) Suppose the arrival rate increases to 2.5 machines per hour. Recalculate the average number of machines in the system and the average waiting time. (6 marks)

(d) Discuss how queuing theory can help improve decision-making in a factory's machine maintenance operations, including scheduling repairs to minimize downtime. (5 marks)

End of Examination